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Asymmetric Systematic Uncertainties in the Determination of Experimental Uncertainty

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A new method is presented for determining a 95% confidence uncertainty interval for an experimental result when some of the measured variables have asymmetric systematic uncertainties. The technique is compared with the approximate method given in the American National Standards Institute/American Society of Mechanical Engineers (ANSI/ASME) standard on measurement uncertainty. Monte Carlo simulations are used to show that the new method provides consistent coverages of about 95% whereas the ANSI/ASME method provides inconsistent confidence intervals.

Nomenclature

B_i = systematic uncertainty of variable i, 95% estimate
$(B_i)_k$ = elemental systematic uncertainty k of variable i, 95% estimate
B_i^- = negative systematic uncertainty of variable i, 95% estimate
B_i^+ = positive systematic uncertainty of variable i, 95% estimate
B_r = systematic uncertainty of result, 95% estimate
B_r^- = negative systematic uncertainty of result, 95% estimate
B_r^+ = positive systematic uncertainty of result, 95% estimate
B_{xy} = correlated systematic uncertainty term
b_i = systematic uncertainty of variable i, standard deviation
b_r = systematic uncertainty of result, standard deviation
b_{xy} = covariance estimator of the systematic error distributions
c_i = shift in mean value μ_i, Eq. (8)
F = asymmetric uncertainty factor, Eq. (29)
h = convective heat transfer coefficient, W/m^{2}K
P = pressure, kPa
P_r = precision uncertainty of result
r = experimental result
r_{true} = true result
S_i = sample standard deviation of variable i
T = temperature, K
T_g = gas temperature, K
T_t = thermocouple temperature, K
T_w = wall temperature, K
t = value from t distribution
U_r = expanded uncertainty of result
u_r = combined standard uncertainty of result
X_i = measured variable i
$X_{i_{true}}$ = true value of variable i
x = measured variable
y = measured variable
β_i = systematic error of variable i
$(\beta_i)_k$ = elemental systematic error k of variable i
γ = specific heat ratio
δ_r = error in result
ϵ = emissivity
ϵ_i = precision error of variable i
η = compressor efficiency
θ_i = sensitivity coefficient for variable i, Eq. (7)
μ_i = parent population mean of measurement i
σ = Stefan–Boltzmann constant, 5.670×10^{-8} W/m^{2}K^4
σ_b = standard deviation of a parent error distribution
$\sigma_{b_{xy}}$ = covariance of the systematic error distributions

Introduction

IN an experimental program, the goal is to answer a question or a group of questions. The ideal answer to a specific question would be r_{true}, but this quantity can never be measured exactly because of experimental uncertainty. Instead, the answer is the experimental result r and the interval around r that contains r_{true} with a specified confidence level. This confidence level is usually taken to be 95% or 99%.

The uncertainty U_r at a 95% confidence level for an experimental result is expressed in terms of a precision uncertainty or precision limit P_r and a systematic uncertainty or bias limit B_r as[1]

$$U_r = \left[B_r^2 + P_r^2\right]^{\frac{1}{2}} \tag{1}$$

The precision interval $\pm P_r$ about a result r (single or averaged) is the experimenter's 95% confidence estimate of the band within which the mean μ of many such results would fall if the experiment were repeated many times under the same conditions and using the same equipment. The precision limit is thus an estimate of the scatter or lack of repeatability caused by random errors and unsteadiness. The bias limit is an estimate of the magnitude of the fixed constant errors. When the true systematic error in a result is defined as β, the quantity B_r is the experimenter's 95% confidence estimate such that $|\beta| \leq B_r$. The total uncertainty interval $\pm U_r$ about the result is

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then the band within which the experimenter is 95% confident that the true value of the result lies.

In this study, we are considering only the systematic uncertainty, specifically, asymmetric systematic uncertainties. The systematic uncertainty, which is our estimate of the true systematic error, is determined using the methods given in Refs. 1 and 2. The systematic error β for a measured variable will be that fixed error that remains after all calibration corrections are made. For a given experiment, the systematic error for each variable will have a single value, but we are never able to measure this value. We always have to estimate it from our best available knowledge of the possible systematic errors.

Systematic errors arise from many sources, such as calibration standard error, calibration curve fit error, the random error in the calibration process that is assumed fixed or fossilized[1] (if multiple calibrations are not performed during the test), conceptual errors caused by the model used for the data reduction equation, and any other fixed error sources. A useful approach to estimating the magnitude of a systematic error is to assume that the error for a given case is a single realization drawn from some statistical parent distribution of possible systematic errors. For example, suppose a thermistor manufacturer specifies that 95% of samples of a given model are within $\pm 1.0^\circ$C of a reference resistance–temperature (R–T) calibration curve supplied with the thermistors. One might assume that the systematic errors (the differences between the actual, but unknown, R–T curves of the various thermistors and the reference curve) result from random processes and, therefore, belong to a Gaussian parent distribution with a standard deviation $b = 0.5^\circ$C. Then the interval defined by $\pm B = \pm 2b = \pm 1.0^\circ$C would include about 95% of the possible systematic errors that could be realized from the parent distribution. If, instead, one assumes that any error value in the distribution has equal probability of occurrence, then a rectangular distribution is appropriate.

Normally, the plus and minus limit estimates for a systematic uncertainty distribution are equal, and for no random uncertainty, we assume that the true value lies somewhere in this symmetric distribution around the reading. There are situations, however, when we have sufficient information to know that the truth is more likely to be on one side of the reading than the other. An example is the radiation error that occurs when a thermocouple is used to measure the temperature of a hot gas flowing in an enclosure with cooler walls. The thermocouple is being heated by convection from the gas, but it is losing heat by radiation to the cooler walls. If the radiation error is a significant component of the systematic uncertainty, then the true gas temperature is more likely to be above the thermocouple temperature reading, and the systematic uncertainty for this measurement would be larger in the positive direction than in the negative direction around the temperature reading.

In some experiments, physical models can be used to replace the asymmetric uncertainties with symmetric uncertainties in additional experimental variables. For instance, performing an energy balance on the thermocouple probe just discussed, the convective heat transfer to the probe is equal to the radiation heat loss to the walls at steady-state conditions assuming negligible conduction loss from the probe

$$h(T_g - T_t) = \epsilon\sigma\left(T_t^4 - T_w^4\right) \tag{2}$$

Solving this equation for T_g yields the new data reduction equation

$$T_g = T_t + (\epsilon\sigma/h)\left(T_t^4 - T_w^4\right) \tag{3}$$

The asymmetric uncertainty in using T_t as the estimate of the gas temperature has been replaced by additional experimental variables that have symmetric uncertainties. This technique might reduce the overall uncertainty in T_g depending on the uncertainties in ϵ, h, and T_w.

When possible, the zero centering of the uncertainty using appropriate physical models as shown should be used when asymmetric systematic uncertainties are present. There may be cases, however, where the additional uncertainties introduced by the additional variables in the data reduction equation yield a larger uncertainty interval than the original asymmetric uncertainty interval or where the experimenter decides that asymmetric uncertainties are more appropriate. In those cases, the methods presented in this study should be used.

In the next section, the uncertainty expression for asymmetric systematic uncertainties is developed. Then a Monte Carlo simulation technique used to qualify the validity of this uncertainty expression is described. Three example experiments are considered for asymmetric systematic uncertainties, each using temperature measurements as the physical model being simulated. The three example experiments are a single temperature measurement, an average of two temperature measurements, and the efficiency of a compressor using temperature measurements at the inlet and at the exit.

Uncertainty Propagation Model

Consider the case where an experimental result r is a function of measured variables x and y

$$r = f(x, y) \tag{4}$$

The systematic uncertainty for each variable is the root sum square of the elemental systematic uncertainties for that variable. These elemental uncertainties arise from separate sources as already discussed. For the case of three elemental systematic uncertainties, B_x is determined as

$$B_x = \left[(B_x)_1^2 + (B_x)_2^2 + (B_x)_3^2\right]^{\frac{1}{2}} \tag{5}$$

B_y is determined in a similar manner. These systematic uncertainties for the variables are then combined using the uncertainty propagation equation[1] as

$$B_r = \left[(\theta_x B_x)^2 + (\theta_y B_y)^2 + 2\theta_x\theta_y B_{xy}\right]^{\frac{1}{2}} \tag{6}$$

where

$$\theta_x = \frac{\partial r}{\partial x}, \qquad \theta_y = \frac{\partial r}{\partial y} \tag{7}$$

and B_{xy} is the sum of the products of elemental systematic uncertainties in x and y that arise from the same source.[3]

The uncertainty expression given by Eq. (6) applies when the systematic uncertainties are symmetric. When the systematic uncertainty interval is asymmetric, as in the case of a radiation error from a thermocouple in a hot-gas stream, the American National Standards Institute/American Society of Mechanical Engineers (ANSI/ASME) standard[2] recommends that separate positive and negative systematic uncertainties for x and y be determined using Eq. (5) to calculate B_x^+, B_x^-, B_y^+, and B_y^- and that these be used to determine an asymmetric uncertainty interval, B_r^+ and B_r^-, using Eq. (6).

This paper presents an alternate method for determining the systematic uncertainty of the result when some of the elemental uncertainties are asymmetric. This method follows the development for symmetric uncertainties given in Appendix B of Ref. 1.

Suppose that the statement of the systematic uncertainty in a variable x is $+B_x^+$ and $-B_x^-$ around the parent population mean of the x measurement μ_x, with a Gaussian uncertainty distribution as shown in Fig. 1; therefore, the truth x_{true} lies in the band $+B_x^+$ and $-B_x^-$ around μ_x. If we add

$$c_x = \frac{B_x^+ - B_x^-}{2} \tag{8}$$

to μ_x to get a modified mean value, then the uncertainty distribution is centered on this modified mean value. We can then use the normal techniques for symmetric systematic uncertainties where the uncertainty band around $\mu_x + c_x$ is the probable location of x_{true} (Fig. 1). The same symmetric band placed around x_{true}, if we knew the exact value of x_{true}, would approximate the systematic error distribution yielding the location of $\mu_x + c_x$ (Fig. 2). Therefore, a specific value of $\mu_x + c_x$ would correspond to a specific systematic error β_{x_i} so that

$$\mu_x + c_x = x_{\text{true}} + \beta_{x_i} \tag{9}$$

Now

$$\mu_x = x_i - \epsilon_{x_i} \tag{10}$$

where ϵ_{x_i} is the precision error. Combining Eqs. (9) and (10) yields

$$x_i + c_x = x_{\text{true}} + \beta_{x_i} + \epsilon_{x_i} \tag{11}$$

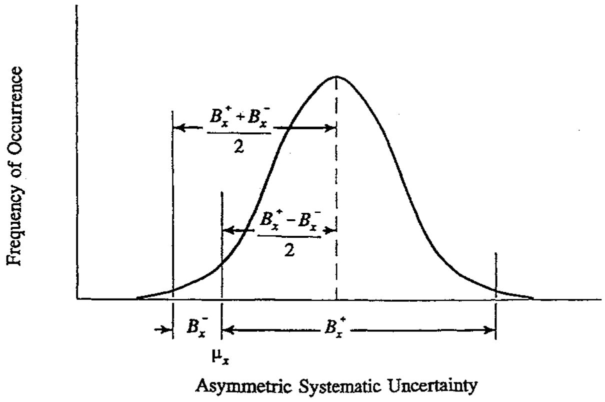

Fig. 1 Asymmetric systematic uncertainty interval around the parent population mean of the measurement x.

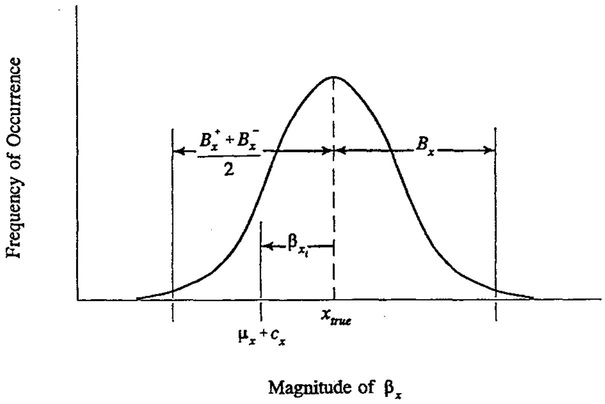

Fig. 2 Systematic error distribution around the true value x_{true} for measurement x.

The distribution for β_{x_i} has a mean of 0 and a 95% confidence band (as shown in Fig. 2) of

$$B_x = \frac{B_x^+ + B_x^-}{2} \tag{12}$$

around x_{true}.

Consider the result that is a function of two variables, $r = r(x, y)$. If both x and y have asymmetric systematic uncertainties and no random uncertainty, $\epsilon_{x_i} = 0$, then

$$x_i + c_x = x_{\text{true}} + \beta_{x_i} \tag{13}$$

and

$$y_i + c_y = y_{\text{true}} + \beta_{y_i} \tag{14}$$

For a first-order Taylor series expansion from $r(x_{\text{true}}, y_{\text{true}})$ to the point $r(x_i + c_x, y_i + c_y)$, we obtain

$$r(x_i + c_x, y_i + c_y) - r(x_{\text{true}}, y_{\text{true}}) = (\theta_x)(x_i + c_x - x_{\text{true}}) + (\theta_y)(y_i + c_y - y_{\text{true}}) \tag{15}$$

where

$$r(x_i + c_x, y_i + c_y) - r(x_{\text{true}}, y_{\text{true}}) = \delta_{r_i} \tag{16}$$

$$x_i + c_x - x_{\text{true}} = \beta_{x_i} \tag{17}$$

$$y_i + c_y - y_{\text{true}} = \beta_{y_i} \tag{18}$$

and where θ_x and θ_y, which should be evaluated at the unknown point $(x_{\text{true}}, y_{\text{true}})$, are instead approximated by evaluating them at $(x_i + c_x, y_i + c_y)$. Then

$$\delta_{r_i} = \theta_x \beta_{x_i} + \theta_y \beta_{y_i} \tag{19}$$

Squaring both sides of Eq. (19), summing over N measurements where it is assumed that β_{x_i} and β_{y_i} vary for each measurement, dividing by N, taking the limit as N approaches infinity, and then taking the square root yields

$$\sigma_{b_r} = \left[\left(\theta_x \sigma_{b_x}\right)^2 + \left(\theta_y \sigma_{b_y}\right)^2 + 2\theta_x \theta_y \sigma_{b_{xy}}\right]^{\frac{1}{2}} \tag{20}$$

where

$$\sigma_{b_x} = \lim_{N\to\infty} \left[\frac{1}{N}\sum_{i=1}^{N} \beta_{x_i}^2\right]^{\frac{1}{2}} \tag{21}$$

$$\sigma_{b_y} = \lim_{N\to\infty} \left[\frac{1}{N}\sum_{i=1}^{N} \beta_{y_i}^2\right]^{\frac{1}{2}} \tag{22}$$

and

$$\sigma_{b_{xy}} = \lim_{N\to\infty} \left[\frac{1}{N}\sum_{i=1}^{N} \beta_{x_i}\beta_{y_i}\right] \tag{23}$$

The quantities σ_{b_x} and σ_{b_y} are the standard deviations of the respective systematic error distributions, and $\sigma_{b_{xy}}$ is the covariance for the systematic errors in x and y. Estimators of these σ are the uncertainties b_x, b_y, and b_{xy}, yielding from Eq. (20)

$$u_r = b_r = \left[(\theta_x b_x)^2 + (\theta_y b_y)^2 + 2\theta_x \theta_y b_{xy}\right]^{\frac{1}{2}} \tag{24}$$

If the systematic error distributions are Gaussian and the uncertainty estimates are based on large degrees of freedom, then the systematic uncertainties B_x and B_y are determined from these standard deviation uncertainties as

$$B_x = 2b_x \tag{25}$$

$$B_y = 2b_y \tag{26}$$

The International Organization for Standardization (ISO) guide[4] calls the uncertainty u_r the combined standard uncertainty. Because random errors were ignored for this derivation, u_r in this case equals the systematic uncertainty of the result b_r at the standard deviation level.

The guide[4] recommends that u_r be multiplied by the appropriate t value from the Student's t distribution at a specified confidence level to obtain an uncertainty estimate at that confidence level. For 95% confidence and for large degrees of freedom for the standard deviation estimates b_x and b_y, t has a value of 2. This t value will provide a 95% confidence level when the systematic uncertainty distributions are Gaussian but might not when other distributions are assumed; however, when multiple elemental error sources act on the experimental variables, then the central limit theorem states that the distribution of results will approach a normal, or Gaussian, distribution. Therefore, the expanded uncertainty[4] U_r and bias limit of the result for 95% confidence, B_r, for this no precision error case are

$$U_r = B_r = 2\left[(\theta_x b_x)^2 + (\theta_y b_y)^2 + 2\theta_x \theta_y b_{xy}\right]^{\frac{1}{2}} \tag{27}$$

This uncertainty interval means that

$$r(x_i + c_x, y_i + c_y) - B_r \le r_{\text{true}} \le r(x_i + c_x, y_i + c_y) + B_r \tag{28}$$

Now if a factor F is defined so that

$$F = r(x_i + c_x, y_i + c_y) - r(x_i, y_i) \tag{29}$$

then

$$r(x_i, y_i) - (B_r - F) \le r_{\text{true}} \le r(x_i, y_i) + (B_r + F) \tag{30}$$

The asymmetric interval $-(B_r - F)$ and $+(B_r + F)$ is the systematic uncertainty for the result $r(x_i, y_i)$.

Assuming Gaussian systematic error distributions and assuming no correlated systematic uncertainties ($b_{xy} = 0$), Eq. (27) becomes

$$B_r = \left[(\theta_x B_x)^2 + (\theta_y B_y)^2\right]^{\frac{1}{2}} \quad (31)$$

where B_x and B_y are the 95% confidence bias limit estimates for the measurements x and y, respectively ($2b_x$ and $2b_y$). B_x and B_y are determined from the elemental systematic uncertainties using Eq. (5) where each elemental asymmetric uncertainty is determined from Eq. (12). If the systematic error distributions are assumed to be rectangular, then Eq. (27) is used to determine B_r, where

$$b_x = \left[\left(\frac{(B_x)_1}{0.95\sqrt{3}}\right)^2 + \left(\frac{(B_x)_2}{0.95\sqrt{3}}\right)^2 + \cdots\right]^{\frac{1}{2}} \quad (32)$$

and

$$b_y = \left[\left(\frac{(B_y)_1}{0.95\sqrt{3}}\right)^2 + \left(\frac{(B_y)_2}{0.95\sqrt{3}}\right)^2 + \cdots\right]^{\frac{1}{2}} \quad (33)$$

The factors $(B_x)_i/(0.95\sqrt{3})$ and $(B_y)_i/(0.95\sqrt{3})$ are used to convert the 95% elemental systematic uncertainty estimates $(B_x)_i$ and $(B_y)_i$ to estimates of the standard deviations for the rectangular distributions. For other assumed asymmetric systematic error distributions, such as a triangular distribution, the value of c_x would be that quantity required to shift μ_x, the mean of the variable x, to the mean of the assumed systematic uncertainty distribution. The elemental systematic uncertainty for that asymmetric error distribution that is combined by root sum square with the other elemental uncertainties for x to determine b_x in Eq. (27) would be the standard deviation of the assumed distribution.

For the general case of

$$r = f(X_1, X_2, \ldots, X_J) \quad (34)$$

Eq. (27) is extended to contain a separate term for each of the J variables X_i and the terms for correlated systematic uncertainties and random uncertainties

$$U_r = 2\left[\sum_{i=1}^{J}(\theta_i b_i)^2 + 2\sum_{i=1}^{J-1}\sum_{k=i+1}^{J}\theta_i\theta_k b_{ik} + \sum_{i=1}^{J}(\theta_i S_i)^2\right]^{\frac{1}{2}} \quad (35)$$

where S_i is the sample standard deviation for variable X_i and where large degrees of freedom[5] have been assumed so that t has a value of 2. In the next section, we describe a computer simulation method used to qualify the asymmetric uncertainty expression given in Eq. (30).

Monte Carlo Simulation

A Monte Carlo simulation technique was used to test the validity of Eq. (30) for predicting the asymmetric uncertainty range around a result that contains the true result. For each example experiment, true values of the independent variables X_i were chosen and used to determine the true value of the experimental result. Also, the parent population distributions (Gaussian or rectangular) were chosen for the elemental systematic errors along with the corresponding systematic uncertainties (bias limits), $(B_i)_k$, for the parent distributions that contained 95% of the possible error values for each systematic error source. For a Gaussian distribution, this input quantity was equal to two times the standard deviation of the distribution. For a rectangular distribution, this 95% limit corrosponded to 95% of the area under the distribution curve. For each variable, three elemental systematic uncertainties were used with one of these having asymmetric components $(B_i)_k^+$ and $(B_i)_k^-$. The other two systematic uncertainties were taken as symmetric. The bias limit for the asymmetric systematic uncertainty was determined using Eq. (12). The three elemental bias limits were combined using Eq. (5) or (32), as appropriate, to obtain the systematic uncertainty for each variable. Random uncertainties were not considered in these example experiments.

For each example case, 10,000 numerical experiments were run. First, for each case, the elemental errors $(\beta_i)_k$ were determined by sampling from a Gaussian or a rectangular random number generator. The value for each variable was then determined from Eq. (13) as

$$X_i = X_{i_{\text{true}}} - c_{X_i} + (\beta_i)_1 + (\beta_i)_2 + (\beta_i)_3 \quad (36)$$

The result was calculated using these X_i values and was also calculated using the $X_i + c_{X_i}$ values in order to determine F from Eq. (29). The value of B_r was determined using Eq. (31) or (27), as appropriate, evaluating the nonconstant derivatives at $X_i + c_{X_i}$, and a check was made to see if r_{true} fell in the interval of $r + (B_r + F)$ and $r - (B_r - F)$. If it did, a counter was incremented for that case. The process was repeated 10,000 times for each example case allowing a coverage fraction to be determined for the asymmetric systematic uncertainty model.

For each example case, the B_r^+ and B_r^- systematic uncertainties specified by the ANSI/ASME standard[2] were also calculated for each iteration. An example of this calculation procedure for a result that is a function of two variables is

$$B_r^+ = \left[\left(\theta_{X_1} B_{X_1}^+\right)^2 + \left(\theta_{X_2} B_{X_2}^+\right)^2\right]^{\frac{1}{2}} \quad (37)$$

and

$$B_r^- = \left[\left(\theta_{X_1} B_{X_1}^-\right)^2 + \left(\theta_{X_2} B_{X_2}^-\right)^2\right]^{\frac{1}{2}} \quad (38)$$

where an example for the bias limit of each variable assuming three elemental systematic uncertainties with one being asymmetric is

$$B_{X_i}^+ = \left\{\left[\left(B_{X_i}\right)_1^+\right]^2 + \left(B_{X_i}\right)_2^2 + \left(B_{X_i}\right)_3^2\right\}^{\frac{1}{2}} \quad (39)$$

and

$$B_{X_i}^- = \left\{\left[\left(B_{X_i}\right)_1^-\right]^2 + \left(B_{X_i}\right)_2^2 + \left(B_{X_i}\right)_3^2\right\}^{\frac{1}{2}} \quad (40)$$

No systematic error distribution specification was made for these $(B_i)_k$ components since the ANSI/ASME standard uses 95% estimates with no reference to the error distribution. For this model, the nonconstant derivatives in Eqs. (37) and (38) were evaluated at the X_i values. A check was made to see if r_{true} fell in the interval of $r + B_r^+$ and $r - B_r^-$. If it did, a counter was incremented for that case so that a coverage fraction for the ANSI/ASME model could be determined.

The variables with uncertainties chosen for each of the example experiments were each taken as a temperature with a radiation asymmetric systematic uncertainty and two symmetric systematic uncertainties. For each example experiment, three variations were run varying both the symmetric and asymmetric systematic uncertainty values.

Example Experiments

Three example experiments were simulated to check the validity of the uncertainty models. These consisted of the measurement of a single temperature, the determination of a mean temperature with measurements from two thermocouples, and the determination of a compressor efficiency from temperature and pressure measurements. For the compressor efficiency case, the pressure was assumed to have no uncertainty for simplicity. The data reduction equations for the three experiments are given in Table 1 along with the true values of the variables and the specified elemental systematic uncertainties. The elemental asymmetric systematic uncertainty for each temperature was taken as $(B_T)_1^- = 1$ K and $(B_T)_1^+ = 1$–50 K. The two elemental symmetric systematic uncertainties were set equal to each other, $(B_T)_2 = (B_T)_3$, and were taken as either 0, 5, or 10 K for the example cases.

The simulation results for two cases of the mean temperature example are given in Figs. 3 and 4. For each case shown, the elemental symmetric systematic uncertainties were taken as 10 K, the elemental negative components of the asymmetric systematic uncertainties were taken as 1 K [$(B_{T_i})_1^- = 1$ K], and 50 separate 10,000-sample Monte Carlo simulations were run varying the elemental positive component of the asymmetric systematic uncertainty from 1 to 50 K

Table 1 Data reduction equations, hypothetical true values of test variables, and elemental systematic uncertainties

Single variable: $T = 300$ K

$$T = 300\text{ K} \qquad (B_T)_1^- = 1\text{ K} \qquad (B_T)_1^+ = 1\text{–}50\text{ K}$$
$$(B_T)_2 = (B_T)_3 = 0, 5, 10\text{ K}$$

Mean of two temperatures: $\bar{T} = (T_1 + T_2)/2 = 305$ K

$$T_1 = 300\text{ K} \qquad (B_{T_1})_1^- = (B_{T_2})_1^- = 1\text{ K} \qquad (B_{T_1})_1^+ = (B_{T_2})_1^+ = 1\text{–}50\text{ K}$$
$$T_2 = 310\text{ K} \qquad (B_{T_1})_2 = (B_{T_2})_2 = (B_{T_1})_3 = (B_{T_2})_3 = 0, 5, 10\text{ K}$$

$$\text{Compressor efficiency: } \eta = \left[\left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]\left[\frac{T_2}{T_1} - 1\right]^{-1} = 0.87$$

$$T_1 = 294\text{ K} \qquad (B_{T_1})_1^- = (B_{T_2})_1^- = 1\text{ K} \qquad (B_{T_1})_1^+ = (B_{T_2})_1^+ = 1\text{–}50\text{ K}$$
$$T_2 = 533\text{ K} \qquad (B_{T_1})_2 = (B_{T_2})_2 = (B_{T_1})_3 = (B_{T_2})_3 = 0, 5, 10\text{ K}$$
$$P_1 = 101\text{ kPa} \qquad B_{P_1} = 0$$
$$P_1 = 650\text{ kPa} \qquad B_{P_2} = 0$$
$$\gamma = 1.4 \qquad B_\gamma = 0$$

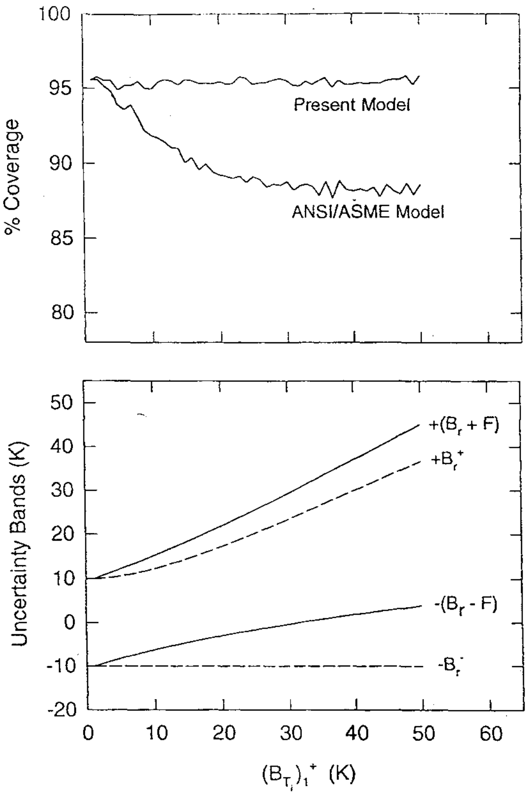

Fig. 3 Simulation results for mean temperature example with Gaussian systematic uncertainties; $(B_{T_i}^-)_1 = 1$ K, $(B_{T_i})_2 = 10$ K, $(B_{T_i})_3 = 10$ K.

$[(B_{T_i})_1^+ = 1\text{–}50$ K]. The results for assumed Gaussian uncertainty distributions are given in Fig. 3 and those for rectangular uncertainty distributions are presented in Fig. 4. For both distributions, the present uncertainty model [Eq. (30)] provides approximately 95% coverage. The ANSI/ASME asymmetric uncertainty model yields coverages that are significantly less than 95%, approaching 88% for Gaussian uncertainty distributions as the positive component of the asymmetric systematic uncertainty increases and approaching 83% for rectangular uncertainty distributions. The uncertainty band for the present model is shifted above the ANSI/ASME predicted band for both cases. It is interesting to note that for the Gaussian uncertainty case (Fig. 3) for every large asymmetric systematic uncertainties, $(B_{T_i})_1^- = 1$ K and $(B_{T_i})_1^-$ 32–50 K, the lower limit of the uncertainty interval is positive [$-(B_r - F)$ is positive]; therefore, the present uncertainty model [Eq. (30)] predicts that the truth lies in an interval that is above the determined result r. At first glance, this situation seems absurd; the uncertainty interval does not contain the experimental result. This is, however, the natural and statistically correct result of the uncertainty model. The absurdity arises from the very large asymmetry. In such cases, the experimenter should consider developing models of the physical phenomena (thermocouple radiation, for example) that caused the asymmetric uncertainty estimate. The simulation results for the other cases for the mean temperature example showed the same trends as those illustrated in Figs. 3 and 4.

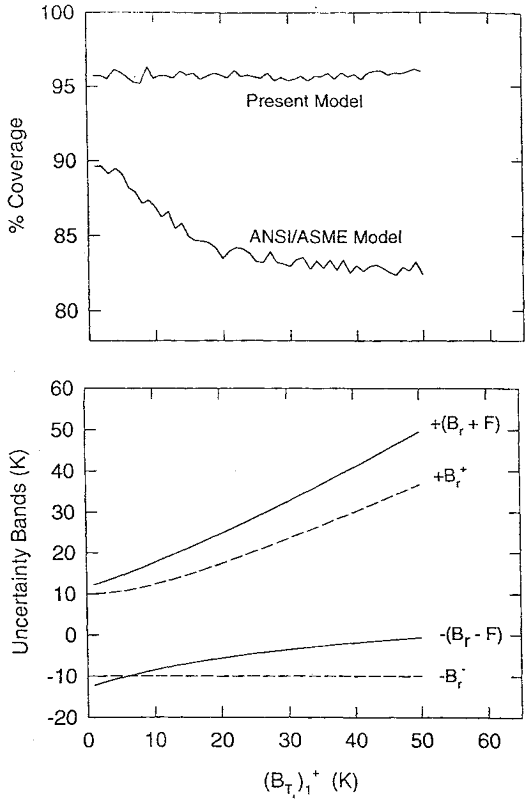

Fig. 4 Simulation results for mean temperature example with rectangular systematic uncertainties; $(B_{T_i}^-)_1 = 1$ K, $(B_{T_i})_2 = 10$ K, $(B_{T_i})_3 = 10$ K.

The results obtained for the compressor efficiency example showed slightly different trends from those discussed for the mean temperature example. For Gaussian systematic uncertainties, the present model provided 95% coverage for all cases. The ANSI/ASME model gave low coverages when the two elemental symmetric systematic uncertainties were zero, but gave coverages that approached 95% as the elemental symmetric systematic uncertainties were set at 5 K and then 10 K. Simulation results for rectangular systematic uncertainties showed consistently low coverages for the ANSI/ASME model. The present model with rectangular systematic uncertainties gave coverages of about 97% for the cases with asymmetric uncertainty only, but approached 95% when there were three nonzero elemental systematic uncertainties for each variable.

For the example of a single measurement of temperature with assumed Gaussian uncertainties, both the present and the ANSI/ASME models gave 95–97% coverages for all cases, with the present model giving consistently 95% coverage. The simulation results for this single measurement example for assumed rectangular systematic uncertainties were different from the other cases. For asymmetric uncertainty only, $(B_T)_2 = (B_T)_3 = 0$, the ANSI/ASME model gave 95% coverage, but the present model gave 100% coverage. The result occurred because there was only one elemental uncertainty, B_x [Eq. (12)]. For the Gaussian cases [Eq. (31)], the present model yielded $B_r = B_x$, but for these rectangular cases [Eqs. (32) and (27)], $B_r = 1.22B_x$. Since B_x was appropriate for 95% coverage, the larger uncertainty interval provided by the rectangular uncertainty distribution provided a higher coverage for the case of asymmetric uncertainty only. When the elemental symmetric systematic

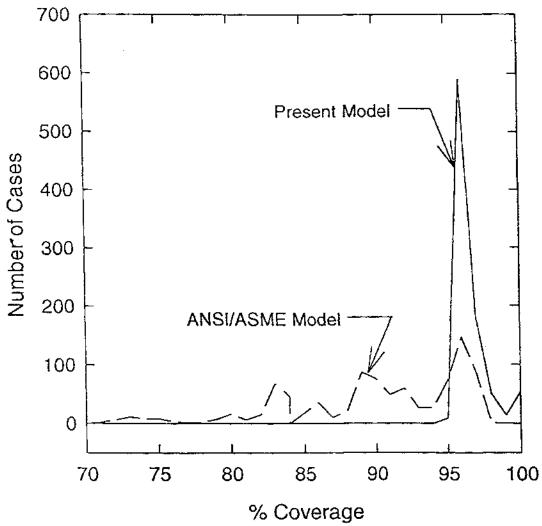

Fig. 5 Histogram of coverages provided by each uncertainty model for all example experiments.

uncertainties for this rectangular case were increased to 5 K, the present model yielded coverages of 96–99%. When the symmetric uncertainties were each taken as 10 K, the present model gave coverages of 96–97%. This approach toward 95% coverage for the rectangular case as more, larger elemental uncertainties were included is an example of the effect of the central limit theorem.

The coverage results from all of the Monte Carlo simulations are shown in Fig. 5. Each curve in the figure represents a histogram of all coverages for all cases considered for each uncertainty model. As seen in this figure, the ANSI/ASME model does not provide any consistency in percent coverage interval prediction. The present model provides 95% coverage for the majority of the cases investigated with a small range of high coverages obtained for the single variable experiment with rectangular systematic uncertainties.

Recommended Procedure

When it is necessary or desirable to use asymmetric systematic uncertainty estimates, the techniques presented in this study should be used to determine the 95% confidence uncertainty interval around the result where it is expected that the true result lies. For each Gaussian or rectangularly distributed asymmetric uncertainty in a measured variable, a factor c_x [Eq. (8)] should be determined along with the equivalent symmetric uncertainty B_x [Eq. (12)]. Then the quantity B_r is determined from Eq. (27) and the factor F from Eq. (29) yielding the asymmetric uncertainty interval for the result $[+(B_r + F)$ and $-(B_r - F)]$.

At first thought, it might seem more reasonable to add the factor c_x to each variable with asymmetric uncertainty and to report the result as $r(x + c_x, y + c_y)$ with a symmetric uncertainty interval; however, this technique is not recommended because c_x is based on an estimated systematic uncertainty and not on a known systematic calibration error. Therefore, it is more correct to report the result determined at the measured variables and the associated asymmetric uncertainty interval.

Conclusions

A technique has been presented for determining a 95% confidence uncertainty interval for an experimental result when some of the measured variables have asymmetric systematic uncertainties. It was shown that this new method provides the desired confidence interval over a wide range of conditions whereas the previously used ANSI/ASME method provides inconsistent confidence intervals. The ANSI/ASME standard[2] is currently being revised, and the new version will recommend using this new method when dealing with asymmetric systematic uncertainties.

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